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FRACTURE MECHANICS OF PIEZOELECTRIC MATERIALS. AXISYMMETRIC CRACK ON THE BOUNDARY WITH A CONDUCTOR

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B. A. KUDRIAVTSEV, V. Z. PARTON and V. I. RAKITIN
(Moscow)

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A condition is formulated which is the generalization of the fracture variational principle in piezoelectric media. In some cases such a representation of the fracture condition, which permits the determination of crack development in a piezoelectric material, turns out to be preferable to the analogous condition obtained in [1].

The problem of a disc-shaped crack developing on the boundary between a piezoelectric ceramic and an elastic isotropic conductor is considered as an illustration.

1. Variational principle of the fracture mechanics of piezoelectric media. The stress components σ_{ij} $(i, j = 1, 2, 3)$ and the components of the electric induction vector of a piezoelectric medium satisfy the equilibrium equations and the Maxwell equation in the statistical case

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad \frac{\partial D_j}{\partial x_j} = 0 \tag{1.1}$$

In Cartesian coordinates referred to the crystal-physics axes, for a piezoelectric medium [2]

$$\sigma_{ij} = c_{ijkl}^E \, \varepsilon_{kl} - e_{ijk} E_k \tag{1.2}$$

$$D_i = e_{kli}\varepsilon_{kl} + \varepsilon_{ik}{}^S E_k \quad (i, j, k, l = 1, 2, 3)$$

Here c_{ijkl}^E are the elastic moduli of the medium, e_{ijk} are the piezoelectric moduli, $\varepsilon_{ik}{}^s$ are the adiabatic dielectric constants, E_k are the electric field strength components, and ε_{kl} are the strain tensor components.

To derive the condition governing crack development in a piezoelectric material, let us examine a number of possible body states just as in [3, 4]. Suppose there is no crack in the body in State 1, and external loads and an electrical potential φ $(E_k = \partial\varphi/\partial x_k)$ is specified on the body surface S. The stresses σ_{ij1}, the displacement vector u_{i1}, the potential φ_1 and the vector of electrical induction D_{j1} correspond to this state. We examine a body in State 2 with the same external loads and potential on the surface S as in State 1, but with a crack Σ, on whose edges a load and potential are specified. The stresses σ_{ij2}, displacements u_{i2}, potential φ_2 and induction D_{j2} correspond to this state. Finally, we examine a body in State 3 with a crack varying in length and shape with the loads and electric field of State 2.

The change in energy during the passage from one State to another is [5]

$$\delta E = -\delta A + \delta W \tag{1.3}$$

Here δA is the work of the external forces and electric field and δW is the change in internal energy of the body.

From the energy conservation law there follows for the passage from State 2 into State 3:

$$\delta U_0 = \delta E_{2-3}, \qquad U_0 = \int_\Sigma \gamma d\Sigma \tag{1.4}$$

Here U_0 is the energy influx associated with the surface energy, and γ is the intensity of the surface fracture energy.

Using the relationship

$$\sigma_{ij2}\varepsilon_{ij1} + E_{j2}D_{j1} = \sigma_{ij1}\varepsilon_{ij2} + E_{j1}D_{j2}$$

which is verified by substitution of (1.2) and by (1.1), it is easy to show that the change in internal energy during passage from State 1 to State 2 is determined by the equation

$$\delta W_{1-2} = \frac{1}{2}\int_V (U_2 - U_1)\,d\tau = \frac{1}{2}\int_{S+\Sigma} (\sigma_{ij2} + \sigma_{ij1})(u_{i2} - u_{i1})\,n_j dS + \tag{1.5}$$

$$\frac{1}{2}\int_{S+\Sigma} (\varphi_2 + \varphi_1)(D_{j2} - D_{j1})\,n_j dS$$

$$U = {}^1/_2\sigma_{ij}\varepsilon_{ij} + {}^1/_2 D_j E_j$$

Here U is the density of the internal energy. The work of the surface forces and the field during passage from State 1 to State 2 can be written as

$$\delta A_{1-2} = \int_S \sigma_{ij1}(u_{i2} - u_{i1})\,n_j dS + \int_S \varphi_1 (D_{j2} - D_{j1})\,n_j dS + \tag{1.6}$$

$$\int_\Sigma (\sigma_{ij2} + \sigma_{ij1})(u_{i2} - u_{i1})\,n_j dS + \int_S (\varphi_2 + \varphi_1)(D_{j2} - D_{j1})\,n_j dS$$

Using (1.3), (1.5), (1.6), we obtain the change in energy for passage from State 1 into State 2

$$\delta E_{1-2} = -\frac{1}{2}\left[\int_\Sigma (\sigma_{ij2} + \sigma_{ij1})(u_{i2} - u_{i1})\,n_j dS + \right. \tag{1.7}$$

$$\left. \int_\Sigma (\varphi_2 + \varphi_1)(D_{j2} - D_{j1})\,n_j dS\right]$$

The change in energy for the passage from State 1 to State 3 can be obtained in an analogous manner

$$\delta E_{1-3} = -\frac{1}{2}\left\{\int_{\Sigma+\delta\Sigma} (\sigma_{ij2} + \sigma_{ij1})[u_{i2} - u_{i1} + \right. \tag{1.8}$$

$$\left. \delta(u_{i2} - u_{i1})]\,n_j dS + \int_{\Sigma+\delta\Sigma} (\varphi_2 + \varphi_1)[D_{j2} - D_{j1} + \delta(D_{j2} - D_{j1})]\,n_j dS\right\}$$

Taking account of the equality

$$\delta E_{2-3} = \delta E_{1-3} - \delta E_{1-2}$$

the fundamental variational relationships governing the crack development condition in a piezoelectric medium

$$\delta \int_{\Sigma} \left\{ \gamma + \frac{1}{2} [(\sigma_{ij2} + \sigma_{ij1})(u_{i2} - u_{i1}) + (\varphi_2 + \varphi_1)(D_{j2} - D_{j1})] n_j \right\} dS = 0 \quad (1.9)$$

follows from (1.4), (1.7), (1.8). If there are no external load and field in State 1, then the condition (1.9) becomes

$$\delta \int_{\Sigma} \left(\gamma - \frac{1}{2} p_i u_i - \frac{1}{2} \varphi D_n \right) dS = 0 \quad (1.10)$$

$$p_i = -\sigma_{ij2} n_j, \quad u_i = u_{i2}, \quad \varphi = \varphi_2, \quad D_n = -D_{j2} n_j$$

Here the positive direction of the normal to the boundary Σ which is external to the medium is taken into account.

2. Axisymmetric crack on the boundary with a conductor. Formulation of the problem. Let us consider an unbounded half-space $(z \geqslant 0)$ of piezoelectric texture which has the symmetry $\infty \cdot m$. The texture is a polycrystalline aggregate consisting of monocrystals whose polarization vector is oriented by the external field. After removal of the field, the polarization vector retains its direction (polarized ceramic).

The disc-shaped crack is located perpendicularly to the polarization direction, which is an axis of symmetry of infinite order for a piezoelectric ceramic on the interface $(z = 0)$ of the piezoelectric medium $(z \geqslant 0)$ and of the elastic isotropic conductor $(z \leqslant 0)$. Let us refer the space to a cylindrical r, θ, z coordinate system so that the z-axis coincides with the axis of symmetry.

The edges of the crack of radius a are loaded by internal pressure $\sigma_0 = \sigma_0(r)$ which is symmetrical relative to the z-axis. Taking account of the symmetry of the load and the properties of the piezoelectric texture under consideration, the equations of the electrical elasticity problem in the r, θ, z coordinate system are [2, 6] $(z \geqslant 0)$:

$$\frac{\partial \sigma_{rr}^+}{\partial r} + \frac{\partial \sigma_{rz}^+}{\partial z} + \frac{\sigma_{rr}^+ - \sigma_{\theta\theta}^+}{r} = 0, \quad \frac{\partial \sigma_{rz}^+}{\partial r} + \frac{\partial \sigma_{zz}^+}{\partial z} + \frac{\sigma_{rz}^+}{r} = 0 \quad (2.1)$$

$$\frac{\partial D_r}{\partial r} + \frac{1}{r} D_r + \frac{\partial D_z}{\partial z} = 0$$

Here $\sigma_{rr}^+, \sigma_{rz}^+, \sigma_{zz}^+$ are the stress tensor components of the piezoelectric medium, and D_r, D_z are components of the electrical induction vector.

Let us select the strain and electric field components as independent variables, and by using the matrix form of writing, let us represent (1.2) as

$$\sigma_i = c_{ij}^E \varepsilon_j - e_{ik} E_k$$

$$D_k = e_{ik} \varepsilon_i + \varepsilon_{kl}^s E_k \qquad (i, j = 1, 2, \ldots, 6;\ k, l = 1, 2, 3) \quad (2.2)$$

$$\sigma_1 = \sigma_{rr}^+, \quad \sigma_2 = \sigma_{\theta\theta}^+, \quad \sigma_3 = \sigma_{zz}^+, \quad \sigma_5 = \sigma_{rz}^+, \quad \sigma_4 = \sigma_6 = 0$$

$$\varepsilon_1 = \frac{\partial u_r^+}{\partial r}, \quad \varepsilon_2 = \frac{u_r^+}{r}, \quad \varepsilon_3 = \frac{\partial u_z^+}{\partial z}, \quad \varepsilon_4 = \varepsilon_6 = 0$$

$$\varepsilon_5 = \frac{\partial u_r^+}{\partial z} + \frac{\partial u_z^+}{\partial r}, \quad D_1 = D_r, \quad D_2 = 0$$

$$D_3 = D_z, \quad E_1 = E_r, \quad E_2 = 0, \quad E_3 = E_z$$

Here u_r^+, u_z^+ are the displacement vector components in the case of axisymmetric strain of the piezoelectric medium. The form of the matrices of the elastic constants c_{ij}^E ($i, j = 1, 2, \ldots, 6$), the piezoelectric moduli e_{ik} ($i = 1, 2, \ldots, 6$; $k = 1, 2, 3$) and the dielectric constants ε_{kl}^s ($k, l = 1, 2, 3$) is presented in [6] for the piezoelectric texture $\infty \cdot m$ (see also [1]).

Let us introduce the electric potential φ

$$E_r = \frac{\partial \varphi}{\partial r}, \qquad E_z = \frac{\partial \varphi}{\partial z}$$

On the basis of (2.2) we obtain relationships connecting the stress and the electric induction vector components to the strain and potential for axisymmetric strain of the medium

$$\sigma_{rr}^+ = c_{11}^E \frac{\partial u_r^+}{\partial r} + c_{12}^E \frac{u_r^+}{r} + c_{13}^E \frac{\partial u_z^+}{\partial z} - e_{31} \frac{\partial \varphi}{\partial z} \tag{2.3}$$

$$\sigma_{\theta\theta}^+ = c_{12}^E \frac{\partial u_r^+}{\partial r} + c_{11}^E \frac{u_r^+}{r} + c_{13}^E \frac{\partial u_z^+}{\partial z} - e_{31} \frac{\partial \varphi}{\partial z}$$

$$\sigma_{zz}^+ = c_{13}^E \left(\frac{\partial u_r^+}{\partial r} + \frac{u_r^+}{r}\right) + c_{33}^E \frac{\partial u_z^+}{\partial z} - e_{33} \frac{\partial \varphi}{\partial z}$$

$$\sigma_{rz}^+ = c_{41}^E \left(\frac{\partial u_r^+}{\partial z} + \frac{\partial u_z^+}{\partial r}\right) - e_{15} \frac{\partial \varphi}{\partial r}$$

$$D_r = e_{15} \left(\frac{\partial u_r^+}{\partial z} + \frac{\partial u_z^+}{\partial r}\right) + \varepsilon_{11}^s \frac{\partial \varphi}{\partial r}$$

$$D_z = e_{31} \left(\frac{\partial u_r^+}{\partial r} + \frac{u_r^+}{r}\right) + e_{33} \frac{\partial u_z}{\partial z} + \varepsilon_{33}^s \frac{\partial \varphi}{\partial z}$$

Substituting (2.3) into (2.1), we obtain the fundamental equations to investigate the axisymmetric strain of a piezoelectric medium (2.4)

$$c_{11}^E \left(\frac{\partial^2 u_r^+}{\partial r^2} + \frac{1}{r}\frac{\partial u_r^+}{\partial r} - \frac{u_r^+}{r^2}\right) + c_{41}^E \frac{\partial^2 u_r^+}{\partial r^2} + (c_{13}^E + c_{44}^E)\frac{\partial^2 u_z^+}{\partial r\, \partial z} - (e_{31} + e_{15})\frac{\partial^2 \varphi}{\partial r\, \partial z} = 0$$

$$c_{41}^E \left(\frac{\partial^2 u_z^+}{\partial r^2} + \frac{1}{r}\frac{\partial u_z^+}{\partial r}\right) + c_{33}^E \frac{\partial^2 u_z^+}{\partial z^2} + (c_{13}^E + c_{44}^E)\frac{\partial}{\partial z}\left(\frac{\partial u_r^+}{\partial r} + \frac{u_r^+}{r}\right) -$$
$$e_{15}\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) - e_{33}\frac{\partial^2 \varphi}{\partial z^2} = 0$$

$$e_{15}\left(\frac{\partial^2 u_z^+}{\partial r^2} + \frac{1}{r}\frac{\partial u_z^+}{\partial r}\right) + e_{33}\frac{\partial^2 u_z^+}{\partial z^2} + (e_{15} + e_{31})\frac{\partial}{\partial z}\left(\frac{\partial u_r^+}{\partial r} + \frac{u_r^+}{r}\right) +$$
$$\varepsilon_{11}^s\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) + \varepsilon_{33}^s \frac{\partial^2 \varphi}{\partial z^2} = 0$$

For an isotropic conducting medium ($z \leqslant 0$) we have

$$\sigma_{rr}^- = (\lambda + 2\mu)\frac{\partial u_r^-}{\partial r} + \lambda\left(\frac{\partial u_z^-}{\partial z} + \frac{u_r^-}{r}\right) \tag{2.5}$$

$$\sigma_{\theta\theta}^- = (\lambda + 2\mu)\frac{u_r^-}{r} + \lambda\left(\frac{\partial u_z^-}{\partial z} + \frac{\partial u_z^-}{\partial r}\right)$$

$$\sigma_{zz}^- = (\lambda + 2\mu)\frac{\partial u_z^-}{\partial z} + \lambda\left(\frac{\partial u_z^-}{\partial r} + \frac{u_r^-}{r}\right)$$

$$\sigma_{rz}^- = \mu\left(\frac{\partial u_r^-}{\partial z} + \frac{\partial u_z^-}{\partial r}\right)$$

Here σ_{rr}^-, σ_{rz}^-, σ_{zz}^-, $\sigma_{\theta\theta}^-$ are the stress tensor components in an isotropic conductor which satisfy (2.1), u_r^-, u_z^- are the displacement vector components in the case of axisymmetric strain of an isotropic medium, and λ, μ are Lamé coefficients.

The following conditions

$$\sigma_{zz}^+(r, 0) = \sigma_{zz}^-(r, 0), \quad \sigma_{rz}^+(r, 0) = \sigma_{rz}^-(r, 0), \ \varphi = 0; \quad 0 \leqslant r < \infty \tag{2.6}$$

$$\sigma_{zz}^+(r, 0) = -\sigma_0(r), \quad \sigma_{rz}^+(r, 0) = 0; \quad 0 \leqslant r < a \tag{2.7}$$

$$u_r^+(r, 0) = u_r^-(r, 0), \quad u_z^+(r, 0) = u_z^-(r, 0); \quad r > a \tag{2.8}$$

must be satisfied in an investigation of the axisymmetric strain of a disc-shaped crack on the flat interface $z = 0$ between two media.

Moreover, $u_r^+ = u_z^+ = u_r^- = u_z^- = \varphi = 0, \quad R = \sqrt{r^2 + z^2} \to \infty$

3. System of dual integral equations. The solution of the system (2.4) will be sought by using the Hankel integral transform

$$u_r^+(r, z) = \int_0^\infty U(z, \xi) J_1(\xi r)\, d\xi, \quad u_z^+(r, z) = \int_0^\infty V(z, \xi) J_0(\xi r)\, d\xi \tag{3.1}$$

$$\varphi(r, z) = \int_0^\infty \Phi(z, \xi) J_0(\xi r)\, d\xi$$

Substituting (3.1) into (2.4), we obtain a system of ordinary differential equations to determine the functions U, V, Φ. Let us write particular solutions of this system for $z \geqslant 0$ which satisfy the conditions at infinity as

$$U = \alpha e^{-k\xi z}, \quad V = \beta e^{-k\xi z}, \quad \Phi = \gamma e^{-k\xi z}$$

Here k are roots with positive real part for the characteristic equation

$$\det \| a_{ij} \| = 0 \tag{3.2}$$

$$a_{11} = c_{44}{}^E k^2 - c_{11}{}^E, \quad a_{12} = -a_{21} = (c_{13}{}^E + c_{44}{}^E) k$$

$$a_{13} = a_{31} = -(e_{31} + e_{15}) k, \quad a_{22} = c_{33}{}^E k^2 - c_{44}{}^E$$

$$a_{23} = -a_{32} = -e_{33} k^2 + e_{15}, \quad a_{33} = \varepsilon_{33}{}^S k^2 - \varepsilon_{11}{}^S$$

An analysis of (3.2) shows that this equation has two real roots $\pm k$, and four complex conjugate roots $\pm\delta \pm i\omega\,(k_1,\ \delta,\ \omega > 0)$ for known piezoceramics. The constants $\alpha(k)$, $\beta(k)$, $\gamma(k)$ which are the solution of the homogeneous system with the matrix (3.2) are determined from the formulas

$$\alpha = a_{12}a_{23} - a_{13}a_{22}, \quad \beta = -a_{11}a_{23} - a_{12}a_{13}, \quad \gamma = a_{11}a_{22} + a_{12}{}^2$$

Therefore, the functions U, V, Φ can be represented as

$$\begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} A_1(\xi)\, e^{-k_1\xi z} + \operatorname{Re}\left\{\begin{bmatrix} \alpha_{21}+i\alpha_{22} \\ \beta_{21}+i\beta_{22} \\ \gamma_{21}+i\gamma_{22} \end{bmatrix} [B_1(\xi)+iC_1(\xi)]\, e^{-\xi z(\delta+i\omega)}\right\} \tag{3.3}$$

$$\alpha_1=\alpha(k_1),\ \ \beta_1=\beta(k_1),\ \ \gamma_1=\gamma(k_1),\ \ \alpha_{21}+\alpha_{22}=\alpha(\delta+i\omega)$$

$$\beta_{21}+i\beta_{22}=\beta(\delta+i\omega),\ \ \gamma_{21}+i\gamma_{22}=\gamma(\delta+t\omega)$$

Taking account of (3.1) and (3.3), we obtain the following expressions for the displacement and potential components:

$$u_r^+(r,\ z)=\int_0^\infty [\alpha_1A_1(\xi)\, e^{-k_1\xi z}+(\alpha_{21}B_1(\xi)-\alpha_{22}C_1(\xi))\, e^{-\delta\xi z}\cos\omega\xi z+ \tag{3.4}$$
$$(\alpha_{22}B_1(\xi)+\alpha_{21}C_1(\xi))\, e^{-\delta\xi z}\sin\omega\xi z]\, J_1(\xi r)\, d\xi$$

$$u_z^+(r,z)=\int_0^\infty [\beta_1A_1(\xi)\, e^{-k_1\xi z}+(\beta_{21}B_1(\xi)-\beta_{22}C_1(\xi))\, e^{-\delta\xi z}\cos\omega\xi z+$$
$$(\beta_{22}B_1(\xi)+\beta_{21}C_1(\xi))\, e^{-\delta\xi z}\sin\omega\xi z]\, J_0(\xi r)\, d\xi$$

$$\varphi(r,z)=\int_0^\infty [\gamma_1A_1(\xi)\, e^{-k_1\xi z}+(\gamma_{21}B_1(\xi)-\gamma_{22}C_1(\xi))\, e^{-\delta\xi z}\cos\omega\xi z+$$
$$(\gamma_{22}B_1(\xi)+\gamma_{21}C_1(\xi))\, e^{-\delta\xi z}\sin\omega\xi z]\, J_0(\xi z)\, d\xi$$

On the basis of (2.3), (3.1) and (3.4), we find

$$u_r^+(r,0)=\int_0^\infty [\alpha_1A_1(\xi)+\alpha_{21}B_1(\xi)-\alpha_{22}C_1(\xi)]\, J_1(\xi r)\, d\xi \tag{3.5}$$

$$u_z^+(r,\ 0)=\int_0^\infty [\beta_1A_1(\xi)+\beta_{21}B_1(\xi)-\beta_{22}C_1(\xi)]\, J_0(\xi r)\, d\xi$$

$$\varphi(r,\ 0)=\int_0^\infty [\gamma_1A_1(\xi)+\gamma_{21}B_1(\xi)-\gamma_{22}C_1(\xi)]\, J_0(\xi r)\, d\xi$$

$$\sigma_{zz}^+(r,\ 0)=\int_0^\infty \left[\frac{m_1}{k_1}A_1(\xi)+\frac{m_2\delta+m_3\omega}{\delta^2+\omega^2}B_1(\xi)-\frac{m_3\delta-m_2\omega}{\delta^2+\omega^2}C_1(\xi)\right]\times$$
$$\xi J_0(\xi r)\, d\xi$$

$$\sigma_{rz}^+(r,0)=\int_0^\infty [m_1A_1(\xi)+m_2B_1(\xi)-m_3C_1(\xi)]\, \xi J_1(\xi r)\, d\xi$$

where we have introduced the notation

$$m_1=e_{15}\gamma_1-c_{44}{}^E\,(k_1\alpha_1+\beta_1)$$
$$m_2=e_{15}\gamma_{21}-c_{44}{}^E\,(\delta\alpha_{21}-\omega\alpha_{22}+\beta_{21})$$
$$m_3=e_{15}\gamma_{22}-c_{44}{}^E\,(\delta\alpha_{22}+\omega\alpha_{21}+\beta_{22})$$

We represent the solution of the equilibrium equations (2.1) for $z\leqslant 0$ as

$$u_r^-(r, z) = \int_0^\infty [A_2(\xi) + B_2(\xi) z\xi] e^{\xi z} J_1(\xi r) d\xi \tag{3.6}$$

$$u_z^-(r, z) = \int_0^\infty \left[-A_2(\xi) + B_2(\xi)\left(\frac{\lambda+3\mu}{\lambda+\mu} - z\xi\right)\right] e^{\xi z} J_0(\xi r) d\xi$$

Using (3.6), we can obtain on the basis of (2.5)

$$u_r^-(r, 0) = \int_0^\infty A_2(\xi) J_1(\xi r) d\xi \tag{3.7}$$

$$u_z^-(r, 0) = \int_0^\infty \left[-A_2(\xi) + B_2(\xi)\frac{\lambda+3\mu}{\lambda+\mu}\right] J_0(\xi r) d\xi$$

$$\sigma_{zz}^-(r, 0) = \int_0^\infty 2\mu\left[-A_2(\xi) + B_2(\xi)\frac{\lambda+2\mu}{\lambda+\mu}\right]\xi J_0(\xi r) d\xi$$

$$\sigma_{rz}(r, 0) = \int_0^\infty 2\mu\left[A_2(\xi) - B_2(\xi)\frac{\mu}{\lambda+\mu}\right]\xi J_1(\xi r) d\xi$$

Satisfying conditions (2.6) on the interface $z = 0$, of the two media, we obtain

$$C_1 = \frac{\gamma_1}{\gamma_{22}} A_1 + \frac{\gamma_{21}}{\gamma_{22}} B_1$$

$$A_2 = \frac{\delta_1(\lambda+2\mu) + \delta_3\mu}{2\mu(\lambda+\mu)\gamma_{22}} A_1 + \frac{\delta_2(\lambda+2\mu) + \delta_4\mu}{2\mu(\lambda+\mu)\gamma_{22}} B_1$$

$$B_2 + \frac{\delta_1+\delta_3}{2\mu\gamma_{22}} A_1 + \frac{\delta_2+\delta_4}{2\mu\gamma_{22}} B_1$$

Here

$$\delta_1 = m_1\gamma_{22} - m_3\gamma_1, \quad \delta_2 = m_2\gamma_{22} - m_3\gamma_{21}$$

$$\delta_3 = \frac{m_1}{k_1}\gamma_{22} - \frac{m_3\delta - m_2\omega}{\delta^2+\omega^2}\gamma_1, \quad \delta_4 = \frac{(m_2\delta + m_3\omega)\gamma_{22} - (m_3\delta - m_2\omega)\gamma_{21}}{\delta^2+\omega^2}$$

We introduce the functions

$$u_z(r) = u_z^+(r, 0) - u_z^-(r, 0), \quad u_r(r) = u_r^+(r, 0) - u_r^-(r, 0)$$

and by satisfying the conditions (2.7), (2.8) we obtain a system of dual integral equations for the functions $A_1(\xi), B_1(\xi)$

$$\sigma_{rz}^+(r, 0) = \frac{1}{\gamma_{22}}\frac{d}{dr}\int_0^\infty [\delta_1 A_1(\xi) + \delta_2 B_1(\xi)] J_0(\xi r) d\xi = 0, \quad 0 \leqslant r < a \tag{3.8}$$

$$\sigma_{zz}^+(r, 0) = \frac{1}{\gamma_{22}}\frac{1}{r}\frac{d}{dr} r\int_0^\infty [\delta_3 A_1(\xi) + \delta_4 B_1(\xi)] J_1(\xi r) d\xi = -\sigma_0(r) \quad 0 \leqslant r < a \tag{3.9}$$

$$u_z(r) = \frac{1}{\gamma_{22}}\int_0^\infty [\delta_5 A_1(\xi) + \delta_6 B_1(\xi)] J_0(\xi r) d\xi = 0, \quad r > a \tag{3.10}$$

$$u_r(r) = \frac{1}{\gamma_{22}}\int_0^\infty [\delta_7 A_1(\xi) + \delta_8 B_1(\xi)] J_1(\xi r) d\xi = 0, \quad r > a \tag{3.11}$$

Here
$$\delta_5 = \beta_1\gamma_{22} - \beta_{22}\gamma_1 - \frac{\delta_1}{2(\lambda+\mu)} - \frac{\delta_3(\lambda+2\mu)}{2\mu(\lambda+\mu)}$$
$$\delta_6 = \beta_{21}\gamma_{22} - \beta_{22}\gamma_{21} - \frac{\delta_2}{2(\lambda+\mu)} - \frac{\delta_4(\lambda+2\mu)}{2\mu(\lambda+\mu)}$$
$$\delta_7 = \alpha_1\gamma_{22} - \alpha_{22}\gamma_1 - \frac{\delta_1(\lambda+2\mu)}{2\mu(\lambda+\mu)} - \frac{\delta_3}{2(\lambda+\mu)}$$
$$\delta_8 = \alpha_{21}\gamma_{22} - \alpha_{22}\gamma_{21} - \frac{\delta_2(\lambda+2\mu)}{2\mu(\lambda+\mu)} - \frac{\delta_4}{2(\lambda+\mu)}$$

4. Solution of the system of dual equations. If the auxiliary functions $p(t)$ and $q(t)$ are introduced by using the relationships

$$\delta_5 A_1(\xi) + \delta_6 B_1(\xi) = \gamma_{22}\int_0^a p(t)\sin\xi t\,dt \tag{4.1}$$
$$\delta_7 A_1(\xi) + \delta_8 B_1(\xi) = \gamma_{22}\int_0^a q(t)\cos\xi t\,dt$$

then by taking account of the values of the integrals [7]

$$\int_0^\infty J_0(\xi r)\sin\xi t\,d\xi = \begin{cases} 0, & t<r \\ \dfrac{1}{\sqrt{t^2-r^2}}, & t>r \end{cases}$$
$$\int_0^\infty J_1(\xi r)\cos\xi t\,d\xi = \begin{cases} \dfrac{1}{r}, & t<r \\ \dfrac{1}{r} - \dfrac{t}{r\sqrt{t^2-r^2}}, & t>r \end{cases}$$

it can be shown that (3.10), (3.11) are satisfied identically. Substituting (4.1) into (3.8), (3.9), and taking into account that [7]

$$\int_0^\infty J_0(\xi r)\cos\xi t\,d\xi = \begin{cases} \dfrac{1}{\sqrt{r^2-t^2}}, & t<r \\ 0, & t>r \end{cases}$$
$$\int_0^\infty J_1(\xi r)\sin\xi t\,d\xi = \begin{cases} \dfrac{t}{r\sqrt{r^2-t^2}}, & t<r \\ 0, & t>r \end{cases}$$

we obtain the following equalities

$$g_{11}\int_r^a \frac{p(t)\,dt}{\sqrt{t^2-r^2}} + g_{12}\int_0^r \frac{q(t)\,dt}{\sqrt{r^2-t^2}} = 0, \quad 0\leqslant r<a \tag{4.2}$$
$$\frac{1}{r}\frac{d}{dr}\left[g_{21}\int_0^r \frac{t\,p(t)\,dt}{\sqrt{r^2-t^2}} + q_{22}\left(\int_0^a q(t)\,dt - \int_r^a \frac{t\,q(t)\,dt}{\sqrt{t^2-r^2}}\right)\right] = \sigma_0(r), \quad 0\leqslant r<a$$

Here

$$g_{11} = \frac{1}{\Delta}(\delta_7\delta_2 - \delta_8\delta_1), \quad g_{12} = \frac{1}{\Delta}(\delta_6\delta_1 - \delta_5\delta_2)$$
$$g_{21} = \frac{1}{\Delta}(\delta_7\delta_4 - \delta_8\delta_3), \quad g_{22} = \frac{1}{\Delta}(\delta_6\delta_3 - \delta_5\delta_4), \quad \Delta = \delta_5\delta_8 - \delta_6\delta_7$$

Integrating (4.2), we obtain a system of two generalized Abel integral equations, say, for the case $\sigma_0(r) = \sigma_0 = \text{const}$

$$g_{11}\int_r^a \frac{p(t)\,dt}{\sqrt{t^2-r^2}} + g_{12}\int_0^r \frac{q(t)\,dt}{\sqrt{r^2-t^2}} = 0, \quad 0 \leqslant r < a \tag{4.3}$$

$$g_{21}\int_0^r \frac{tp(t)\,dt}{\sqrt{r^2-t^2}} + g_{22}\left(\int_0^a q(t)\,dt - \int_r^a \frac{tq(t)\,dt}{\sqrt{t^2-r^2}}\right) = \sigma_0\frac{r^2}{2}, \quad 0 \leqslant r < a \tag{4.4}$$

Applying the operator

$$\int_0^r \frac{r(\ldots)\,dr}{\sqrt{t^2-r^2}}$$

to (4.3) and changing the order of integration according to the Dirichlet formula, we find

$$\frac{\pi}{2}\int_0^t q(\tau)\,d\tau = -\frac{g_{11}}{g_{12}}\left[\frac{1}{2}\int_0^t p(\tau)\ln\frac{t+\tau}{t-\tau}\,d\tau + \frac{1}{2}\int_t^a p(\tau)\ln\frac{\tau+t}{\tau-t}\,d\tau\right] \tag{4.5}$$

Combining the integrals in the right side of (4.5) and differentiating this equality with respect to t, we obtain

$$q(t) = \frac{2}{\pi}\frac{g_{11}}{g_{12}}\int_0^a \frac{p(\tau)\,\tau\,d\tau}{t^2-\tau^2} \tag{4.6}$$

It can be shown in an analogous manner that (4.4) after appropriate transformations, becomes

$$p(t) = -\frac{2}{\pi}\frac{g_{22}}{g_{21}}t\int_0^a \frac{q(\tau)\,d\tau}{t^2-\tau^2} + \frac{2}{\pi}\frac{\sigma_0 t}{g_{21}} \tag{4.7}$$

Setting $q(-\tau) = q(\tau)$ and $p(-\tau) = -p(\tau)$, then (4.6), (4.7) can be written as a system of singular integral equations with Cauchy kernels

$$p(t) = \frac{g_{22}}{g_{21}}\frac{1}{\pi}\int_{-a}^{a} \frac{q(\tau)\,d\tau}{\tau-t} + \frac{2}{\pi}\frac{\sigma_0}{g_{21}}t \tag{4.8}$$

$$q(t) = -\frac{g_{11}}{g_{12}}\frac{1}{\pi}\int_{-a}^{a} \frac{p(\tau)\,d\tau}{\tau-t} \tag{4.9}$$

Multiplying (4.8) by $i\,/\,g_1$ and adding to (4.9), we obtain a single integral equation

$$f(t) + \frac{1}{g}\frac{1}{\pi i}\int_{-a}^{a} \frac{f(\tau)\,d\tau}{\tau-t} = \frac{2}{\pi}\frac{\sigma_0}{g_1 g_{21}}t \tag{4.10}$$

$$f(t) = q(t) + i\frac{1}{g_1}p(t), \quad g = \frac{g_{21}}{g_{22}}g_1, \quad g_1 = \left(\frac{g_{22}g_{12}}{g_{11}g_{21}}\right)^{1/2}$$

We note that for real-piezoceramics and elastic media, say [1]

$$g_{22}\,/\,g_{21} > 0, \quad g_{11}\,/\,g_{12} > 0, \quad g > 1$$

Following [8], let us introduce the function

$$F(z) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{f(\tau)\,d\tau}{\tau-z}$$

for the solution of (4.10), which is analytic in a plane with a slit along the segment

$-a \leqslant x < a$ of the real axis. Then

$$f(t) = F^+(t) - F^-(t) \tag{4.11}$$

and (4.10) is equivalent to a Riemann boundary value problem

$$F^+(t) - \left(\frac{g-1}{g+1}\right) F^-(t) = \frac{2}{\pi} \frac{i}{g+1} \frac{\sigma_0}{g_{22}} t \tag{4.12}$$

Let us determine the solution which remains bounded near the ends $\pm a$ and let us take the canonical function in the form [8]

$$X(z) = \left(\frac{z-a}{z+a}\right)^{i\varkappa}, \quad \varkappa = \frac{1}{2\pi} \ln \frac{g+1}{g-1}$$

Then the genral solution of the boundary value problem (4.12) which satisfies the condition $F(\infty) = 0$ is given by the formula

$$F(z) = \frac{2}{\pi} \frac{1}{g+1} \frac{X(z)}{2\pi i} \int_{-a}^{a} \frac{i\sigma_0 t\, dt}{g_{22} X^+(t)(t-z)} \tag{4.13}$$

$$X^+(t) = e^{-\pi\varkappa} \left(\frac{a-t}{a+t}\right)^{i\varkappa}$$

Here $X^+(t)$ is the value of $X(z)$ on the upper edge of the slit. Evaluating the integral in (4.13) and using (4.11), we find

$$F(z) = \frac{i\sigma_0}{\pi g_{22}} z + \left(\frac{2\varkappa a\sigma_0}{\pi g_{22}} - \frac{i\sigma_0 z}{\pi g_{22}}\right) X(z) \tag{4.14}$$

$$f(t) = \frac{2}{\pi(g-1)} \left(i \frac{\sigma_0}{g_{22}} t - \frac{\sigma_0}{g_{22}} 2\varkappa a\right) X^+(t)$$

Separating real and imaginary parts in (4.14), we obtain

$$q(t) = -\frac{2}{\pi} \frac{\sigma_0}{g_{22}\sqrt{g^2-1}} \left[t \sin\left(\varkappa \ln \frac{a-t}{a+t}\right) + 2\varkappa a \cos\left(\varkappa \ln \frac{a-t}{a+t}\right)\right]$$

$$p(t) = \frac{2}{\pi} \frac{g_1\sigma_0}{g_{22}\sqrt{g^2-1}} \left[t \cos\left(\varkappa \ln \frac{a-t}{a+t}\right) - 2\varkappa a \sin\left(\varkappa \ln \frac{a-t}{a+t}\right)\right]$$

Using these latter relationships, we can determine all the strain, stress and electric field components in the neighborhood of the disc-shaped crack. In particular, we obtain for the displacement of the edges of the crack and the normal stresses on the continuation of the slit

$$u_z^+(r,0) - u_z^-(r,0) = u_z(r) = \int_r^a \frac{p(t)\,dt}{\sqrt{t^2-r^2}}, \quad 0 \leqslant r < a$$

$$\sigma_{zz}(r,0) = -g_{21} \frac{1}{r} \frac{d}{dr} \int_0^a \frac{p(t)\, t\, dt}{\sqrt{r^2-t^2}}, \qquad r > a$$

Let us use condition (1.10), which becomes in the case under consideration

$$\gamma = \frac{\sigma_0}{2a} \frac{d}{da} \int_0^a r \int_r^a \frac{p(t)\,dt}{\sqrt{t^2-r^2}}\, dr \tag{4.15}$$

in order to determine the magnitude of the critical load acting on the edges of a disc crack. Inverting the order of integration in (4.15), we obtain

$$\gamma = \frac{\sigma_0}{2a}\frac{d}{da}\int_0^a p(t)\, t\, dt$$

It can be shown that

$$\int_0^a p(t)\, t\, dt = \frac{g_1\sigma_0}{g_{22}}\frac{2}{3}\varkappa a^3(1+4\varkappa^2)$$

Then the value of the critical load applied to the crack edges is

$$\sigma_0 = \sqrt{\frac{\gamma g_{21}}{ag\varkappa(1+4\varkappa^2)}}$$

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ON THE ESCAPE PROBLEM WITH CONSTRAINTS OF DIFFERENT TYPES

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A. V. MEZENTSEV
(Moscow)

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We examine a linear escape problem in which the pursuing player's control is constrained in energy, while that of the escaping player, in absolute value. The game's termination set is defined as the equality of the players' geometric coordinates. We have obtained sufficient conditions for the possibility of evasion from contact from any point of the phase space, not belonging to the game's termination set, and sufficient conditions for the existence of an open set in the phase space, from any point of which the game can be terminated in finite time.

Suppose that in the space R^n $(n \geqslant 2)$ the motion of the pursuing vector x and of the escaping vector y is described by the equations